Homework 5 Solutions

Math 131B-2

- (4.11) a) $\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\frac{x^2}{x^2} = 1$. $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\frac{-y^2}{y^2} = -1$. $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

  b)$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\frac{0}{x^2} = 0$. $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\frac{0}{y^2} = 0$. $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, because taking the limit of f evaluated on a sequence of points on the line $x = y$ gives 1, not 0.

  c) $\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} 0 = 0$. $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} y = 0$. [Use L'Hopital's Rule to evaluate the limit of $\frac{\sin(xy)}{x}$ as $x \to 0$.]

  For the limit of the function, observe that for any sequence of points $(x_n, y_n) \to 0$ and any $\epsilon > 0$, we can find N such that $n \geq N$ implies that $|x_n < \frac{\pi}{2}|$ and $|y_n| < \epsilon$. Then if $x_n \neq 0$, since sin is increasing on $[0, \frac{\pi}{2}]$, for $n > N$ we have $|\frac{\sin(x_n y_n)}{x_n}| < |\frac{\sin(\epsilon \cdot x_n)}{x_n}|$ and the limit of the righthand side is ϵ as $x_n \to 0$. If $x_n = 0$, for $n > N$ we have $|f(x,y)| = |y| < \epsilon$. So the limit of $|\frac{\sin(x_n y_n)}{x_n}|$ as $n \to \infty$ is less than or equal to ϵ for any $\epsilon > 0$, hence is 0.

  d) The limit $\lim_{x\to 0} f(x,y)$ does not exist for $y \neq 0$, so the limit $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ does not exist. Similarly, the limit $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ does not exist. However, $|f(x,y)| \leq |x+y|$ for all (x,y), so since $\lim_{(x,y)\to(0,0)} |x+y| = 0$, $\lim_{(x,y)\to(0,0)} = 0$.

- (4.28) (a) Consider $f(x) = \sin(\pi x)$. (b) No such function, because S is connected and T is not. (c) No such function, because S is connected and T is not. (d) Consider f such that $f(x) = 0$ for $x \in [0,1]$ and $f(x) = 1$ for $x \in [2,3]$. (e) No such function, because S is compact and T is not. (f) No such function, because S is compact and T is not. (g) Consider $f(x,y) = (\tan(\frac{\pi}{2}(x - \frac{1}{2})), \tan(\frac{\pi}{2}(y - \frac{1}{2})))$. There are of course other possible examples for (a) and (f).

- (4.37) If S is not connected, then we can write $S = A \cup B$ such that A and B are open, nonempty, and disjoint. Since $B = A^c$ and $A = B^c$, A and B are also closed, and neither can be the empty set or the entire space. Ergo S contains sets which are both open and closed and not equal to either S or $\emptyset$. Conversely, if S contains a set A which is open and closed and not S or $\emptyset$, then $B = A^c$ is open and nonempty, and $S = A \cup B$. So S is disconnected. The problem statement follows.

- (4.49) Suppose that for some $a \in S$ and $r > 0$, the set $\{x : d(x, a) < r\}$ is empty. Then let $A = B(x; r)$ and let $B = \{x : d(x, a) > r\}$. Then A is open, and $B = S - \overline{B}(a; r)$ is also open. Furthermore, A is nonempty because $a \in A$, and B is nonempty because S is unbounded. Moreover, $A \cap B = \emptyset$, and $S = A \cup B$. Since S was assumed to be connected, this is impossible, so we obtain a contradiction.

- (4.52) Let $f : S \to T$ be uniformly continuous and $S \subset \mathbb{R}^n$ be bounded. Then $\overline{S} \subset \mathbb{R}^n$ is closed and bounded, hence compact. Let $\epsilon = 1$, and choose δ such that $||x - y|| < \delta$ implies that $d_T(f(x), f(y)) < 1$. The collection $\{B(x; \delta) : x \in S\}$ is an open covering of $\overline{S}$ (we don't need to include balls centered at points of $\overline{S}$ not in S because there any point of S' is at distance less than ϵ for some x in S). Ergo by compactness there is a finite subcover $B(x_1; \delta), \cdots, B(x_n; \delta)$ which covers $\overline{S}$, and hence also covers S. Since each $x_i \in S$, we conclude that $f(S) \subset \bigcup_{i=1}^{\infty} f(S \cap B(x_i; \delta)) \subset \bigcup_{i=1}^{\infty} B(f(x_i); 1)$. This last set is a finite union of bounded sets, hence certainly bounded: for example, $f(S)$ is contained in $B(f(x_1); r)$ where $r = \max\{d_T(f(x_1), f(x_2)) + \epsilon, \cdots, d_T(f(x_1), f(x_n)) + \epsilon, \epsilon\}$.

  Note: This result is not true for arbitrary metric spaces. Consider the map from the natural numbers with the discrete metric to the natural numbers with the metric they inherit as a subset of the real numbers which takes each n to itself. This is trivially uniformly continuous (take $\delta = \frac{1}{2}$ for any $\epsilon > 0$) but it maps a bounded set to an unbounded set.

- (4.54) Let $f : S \to T$ be uniformly continuous, and $\{x_n\}$ be a Cauchy sequence in S. Then given $\epsilon > 0$, there exists $\delta > 0$ such that for $x, y \in S$, $d_S(x, y) < \delta$ implies that $d_T(f(x), f(y)) < \epsilon$. Moreover, there exists N such that $n, m > N$ implies $|x_n - x_m| < \delta$. Ergo $n, m > N$ implies $d_T(f(x_m), f(x_n)) < \epsilon$. Since ϵ was arbitrary, $\{f(x_n)\}$ is Cauchy.

- (Question 4)
  - Notice $f_n(0) = 0 = f_n(1)$ for all n. For $x \neq 0, 1$, an application of L'Hopital's rule to $f_n(x) = \frac{nx}{(1-x)^{-n}}$ shows that the limit as $n \to \infty$ is 0.
  - Since $|f_n(x)| < \frac{1}{\sqrt{n+1}}$, this sequence converges pointwise to 0 on $[0, 1]$.
  - Since $f_n(\pi) = (-1)^n$ for all n, we conclude this sequence of functions does not converge pointwise on $[0, 2\pi]$.
  - Since $f_n(1) = n^2$ for all n, this sequence does not converge pointwise on $[0, 1]$.

- (Question 5)(a) It's clear that $f_n(x)$ converges to f such that $f(x) = 0$ whenever $x \neq 1$ and $f(1) = 1$. Since each f_n is continuous and the limit f is not continuous, we conclude the convergence cannot be uniform.

  (b)We claim the series converges uniformly to the zero function. Let $\epsilon > 0$. Because g is continuous and $g(1) = 0$, there is some $(1 - \delta, 1]$ on which $|g(x)| < \epsilon$. Therefore since

$|f_n(x)| < 1$ for all n and all $x \in [0,1]$, on $(1-\delta, 1]$, $|g(x) \cdot f_n(x)| < \epsilon$. Now, because g is continuous on $[0,1]$, g is bounded on $[0,1]$, so there is M such that $|g(x)| \leq M$ on the interval. Choose N such that $M \cdot (1-\delta)^N < \epsilon$. Then for any $x \in [0, 1-\delta]$, if $n > N$, $|(g(x) \cdot f_n(x)| \leq M|x^n| \leq M|x^N| \leq M(1-\delta)^N < \epsilon$. We conclude that for any $x \in [0,1]$ and all $n > N$, we have $|g(x) \cdot f_n(x) - 0| < \epsilon$. Ergo the sequence converges uniformly to the zero function.

This sort of interval-splitting will become an important tool as we continue to study convergence of functions.

- (Question 6) Suppose $f_n : S \to \mathbb{R}$ is a sequence of functions, $f_n \to f$ uniformly and each f_n is bounded. Because $\{f_n\}$ satisfies the Cauchy criterion, for $\epsilon = 1$, there exists some N such that $n, m \geq N$ implies that $|f_n(x) - f_m(x)| < 1$ for all $x \in X$. Consider f_N. We know f_N is bounded, so there is some M such that $|f_N(x)| < M$ for all $x \in X$. Therefore for $m > N$, $|f_n(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| < \epsilon + M$. Now, for $k < N$, let M_k be a bound on $|f_k(x)|$. Then if we let $M' = \max\{M + \epsilon, M_1, \cdots, M_k\}$, we see that $|f_n(x)| < M'$ for any $n \in \mathbb{N}$ and $x \in X$.